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The Design of Finite Impulse Response Digital Filters Using Linear Programming Techniques

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In this paper it is shown how standard linear programming techniques can be applied to designing finite impulse response digital filters. Attention is concentrated on designing filters having exactly linear phase, and arbitrary magnitude response. The design method is illustrated by examples of the design of frequency sampling filters with constraints on in-band ripple, optimal filters where the passband and stopband cutoff frequencies may be specified exactly, and filters with simultaneous constraints on the time response and frequency response.

I. INTRODUCTION

Many techniques exist for designing digital filters using optimization procedures. Herrmann and Schuessler have designed equiripple error approximations to finite impulse response (FIR) lowpass and bandpass filters using nonlinear programming procedures.[1,2] This work has been extended by Hofstetter, Oppenheim, and Siegel,[3] and by Parks and McClellan[4] to solve for the desired filters using polynomial interpolation techniques. Rabiner, Gold, and McGonegal[5] used a steepest descent technique to obtain FIR digital filters with minimax error in selected bands with the constraint that only a few of the filter coefficients were variable. Steiglitz,[6] and Athanasopoulos and Kaiser[7] have used nonlinear optimization techniques to obtain recursive filter approximations to arbitrary frequency response specifications.

Recently, attention has been focused on the use of linear programming techniques for the design of digital filters.[8–10] Many digital filter design problems are inherently linear in the design parameters, and hence are natural candidates for linear programming optimization. Further, linear programs are easy to implement and are generally guaranteed

to converge to a unique solution. The rate of convergence of the programs is moderately fast, tbus making tbis technique practical for problems with 100 parameters or so.

There are many areas of FIR filter design where linear programming can be used conveniently. These include:

(*i*) design of filters with minimax ripple in the passband and/or stopband;

(*ii*) design of optimal (minimax) absolute or relative error approximations to arbitrary frequency response characteristics, where the passband and stopband edge frequencies of the filter may be specified exactly;

(*iii*) design of two-dimensional filters of the frequency sampling type, or with optimal error approximation;

(*iv*) and design of filters with simultaneous constraints on characteristics of both the time and frequency response of the filter.

Several of these design areas have been examined and examples will be presented showing how to apply linear programming techniques in specific cases. In the next section, the general framework of linear programming is presented and several practical aspects of linear programs are discussed. The following sections show how the general FIR, linear phase, filter design problem is linear in either the filter impulse response coefficients, or equivalently the Discrete Fourier Transform (DFT) coefficients, and how this problem is solved in specific cases.

II. LINEAR PROGRAMMING

The general linear programming problem can be mathematically stated in the form: find $\{X_j\}$, $j = 1, 2, \cdots, N$ subject to the constraints:

$$X_j \geqq 0, \qquad j = 1, 2, \cdots, N; \tag{1}$$

$$\sum_{j=1}^{N} c_{ij}X_j = b_i\,, \qquad i = 1, 2, \cdots, M(M < N); \tag{2}$$

sucb tbat:

$$\sum_{j=1}^{N} a_jX_j \quad \text{is minimized.} \tag{3}$$

Tbe above problem is referred to as the "primal problem" and by a duality principle can be shown to be matbematically equivalent to

the "dual problem": find $\{Y_i\}$, $i = 1, 2, \cdots, M$ subject to the constraints:

$$\sum_{i=1}^{M} c_{ij} Y_i \leqq a_j \,, \qquad j = 1, 2, \cdots, N, \tag{4}$$

such that:

$$\sum_{i=1}^{M} b_i Y_i \quad \text{is maximized.} \tag{5}$$

The remainder of this paper refers to the dual problem as this is the most natural form for the digital filter design problems under consideration.

One characteristic of linear programs is that, given there is a solution, it is guaranteed to be a unique solution; and there are several well defined procedures for arriving at this solution within $(M + N)$ iterations. There are also straight-forward techniques for determining if the solution is unconstrained or poorly constrained.

The next section shows that linear phase FIR filters are linear in the design parameters and hence can be optimally designed using linear programming techniques.

III. LINEAR PHASE FIR FILTERS

Let $\{h_n\}$, $n = 0, 1, \cdots, N - 1$ be the impulse response of a causal FIR digital filter. The requirement of linear phase implies that

$$h_n = h_{N-n-1} \,. \tag{6}$$

The filter frequency response can be determined, in terms of the $\{h_n\}$, as:

$$H(e^{i\omega T}) = \sum_{n=0}^{N-1} h_n e^{-i\omega T n}. \tag{7}$$

For the case where N is odd, eq. (7) can be combined with eq. (6) to give:

$$H(e^{i\omega T}) = \underbrace{e^{-i\omega((N-1)/2)T}}_{\substack{\text{linear phase}\\ \text{term}}} \cdot \underbrace{\left[h_{(N-1)/2} + \sum_{n=0}^{(N-3)/2} 2h_n \cos\left[\left(\left(\frac{N-1}{2}\right) - n\right)\omega T\right]\right]}_{\substack{\text{purely real}\\ \text{linear in } \{h_n\}}}. \tag{8}$$

Equation (8) shows $H(e^{j\omega T})$ to consist of a purely linear phase term corresponding to a delay of $((N-1)/2)$ samples, and a term which is purely real and linear in the impulse response coefficients. It is the second term in eq. (8) which is used for approximating arbitrary magnitude response characteristics. Where N is even, the result of eq. (8) is modified to:

$$H(e^{j\omega T}) = \underbrace{e^{-j\omega((N-1)/2)T}}_{\substack{\text{linear phase}\\ \text{term}}}\underbrace{\left[\sum_{n=0}^{((N/2)-1)} 2h_n \cos\left(\frac{N-1}{2} - n\right)\omega T\right]}_{\substack{\text{purely real}\\ \text{linear in } \{h_n\}}}. \tag{9}$$

Equation (9) shows that for N even, the linear phase term corresponds to a delay of an (integer $+\frac{1}{2}$) number of samples. The center of symmetry of $\{h_n\}$ is midway between samples $(N/2)$ and $(N/2-1)$. The remainder of eq. (9) is again a real term which is linear in the impulse response coefficients.

The DFT relation can be used to show that the filter frequency response is also a linear function of the DFT coefficients $\{H_k\}$. It is derived elsewhere[11] that the frequency response of linear phase FIR filters can be written:

$$H(e^{j\omega T}) = e^{-j\omega T((N-1)/2)}\,\frac{\sin\dfrac{\omega NT}{2}}{N}$$

$$\cdot\left[\frac{H_0}{\sin\dfrac{\omega T}{2}} - \sum_{k=1}^{K} \frac{(-1)^k H_k 4\cos\dfrac{\pi k}{N}\sin\dfrac{\omega T}{2}}{\left(\cos\omega T - \cos\dfrac{2\pi k}{N}\right)}\right], \tag{10}$$

when

$$K = \begin{cases} (N-1)/2 & \text{for } N \text{ odd} \\ \dfrac{N}{2} - 1 & \text{for } N \text{ even} \end{cases}.$$

The significance of eq. (10) is that the frequency response of a linear phase FIR filter is linear in the $\{H_k\}$ as well as in the $\{h_n\}$; hence linear programming techniques can be used to optimize the values of all or a selected set of DFT or impulse response coefficients.

IV. DESIGN OF FREQUENCY SAMPLING FILTERS

Previously, design of frequency sampling filters was accomplished using a steepest descent minimization.[5] This tecbnique was capable only of minimizing the peak out-of-band ripple when several DFT coefficients in a transition band between passbands and stopbands were varied. Another limitation of the tecbnique was that the amount of computation it took to optimally choose the variable DFT coefficients grew exponentially witb the number of unconstrained variables. The largest problems attempted had four coefficients variable. Tbis problem is readily solved in a much more general form using linear programming techniques. Furtbermore, the computation required to calculate the more general solutions is considerably less tban for the steepest descent algorithm used previously.

A typical specification for a lowpass filter to be approximated by a frequency sampling design is sbown in Fig. 1. Tbe heavy points show the DFT coefficients, and the solid curve shows the interpolated frequency response. Tbe passband edge frequency is F_1 and the stopband edge frequency is F_2 . Since the length of tbe filter impulse response

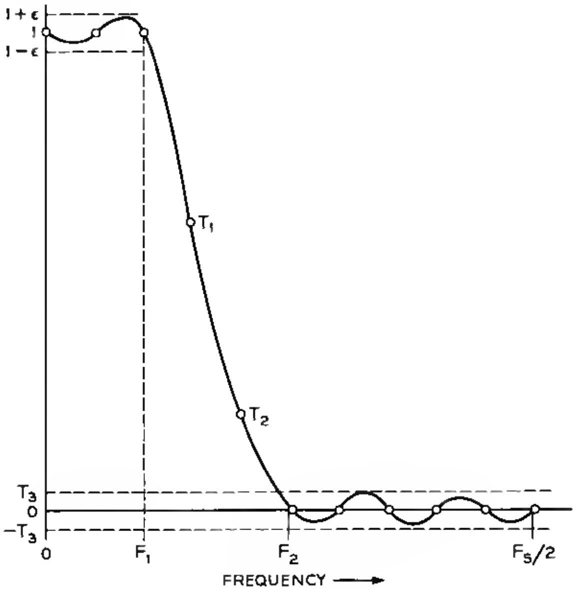

Fig. 1—Typical specification for a frequency sampling lowpass filter with transition coefficients T_1 , T_2 .

is N samples (assume N even), there are $(N/2 + 1)$ DFT coefficients (called frequency samples) to be specified. Those DFT coefficients which are in the passband are arbitrarily assigned the value 1.0, and those that fall in the stopband are assigned the value 0.0. The DFT coefficients in the transition band are free variables, and are labeled T_1, T_2 in Fig. 1. The approximation problem can be set up as a linear program in the following manner. We let

$$T_3 = \text{peak stopband ripple.}$$

Then the design problem consists of finding values of (T_1, T_2) to satisfy the constraints:

(*i*) The in-band ripple is less than or equal to some prescribed tolerance, ϵ.

(*ii*) The peak out-of-band ripple, T_3 is to be minimized.

Mathematically this problem can be stated as: find (T_1, T_2, T_3) subject to the constraints:

$$1 - \epsilon \leqq F(\omega) + \sum_{i=1}^{2} T_i D(\omega, i) \leqq 1 + \epsilon, \qquad 0 \leqq \omega \leqq 2\pi F_1, \tag{11}$$

$$-T_3 \leqq F(\omega) + \sum_{i=1}^{2} T_i D(\omega, i) \leqq T_3, \qquad 2\pi F_2 \leqq \omega \leqq \pi F_s, \tag{12}$$

where $F(\omega)$ is the contribution of the fixed DFT coefficients (the 1.0's in-band) and $D(\omega, i)$ is the contribution of the ith variable transition coefficient and is of the form shown in eq. (10), and F_s is the sampling frequency.

A suitable reshuffling of terms in eqs. (11) and (12) puts the set of equations in the form of the dual problem of linear programming. The final equations are of the form: find (T_1, T_2, T_3) subject to the constraints:

$$\left.\begin{aligned} \sum_{i=1}^{2} T_i D(\omega, i) &\leqq 1 + \epsilon - F(\omega) \\ -\sum_{i=1}^{2} T_i D(\omega, i) &\leqq -1 + \epsilon + F(\omega) \end{aligned}\right\} 0 \leqq \omega \leqq 2\pi F_1, \tag{13}$$

$$\left.\begin{aligned} \sum_{i=1}^{2} T_i D(\omega, i) - T_3 &\leqq -F(\omega) \\ -\sum_{i=1}^{2} T_i D(\omega, i) - T_3 &\leqq F(\omega) \end{aligned}\right\} 2\pi F_2 \leqq \omega \leqq \pi F_s, \tag{14}$$

$(-T_3)$ is maximized.

The inequalities of eqs. (13) and (14) are evaluated at a dense set of frequencies in the appropriate range of interest (an 8-1 interpolation between DFT coefficients is sufficient) to yield the necessary set of equations for the linear program.

V. RESULTS ON FREQUENCY SAMPLING DESIGNS

A wide variety of frequency sampling filters has been designed using the results of eqs. (13) and (14). Previously, using the steepest descent algorithm, constraints on the in-band ripple, ϵ, could not be maintained.[5] With the linear programming design, tradeoff relations between in-band and out-of-band ripple can be obtained for a fixed number of transition samples, or equivalently a fixed width of transition band. Such tradeoff relations are illustrated in Figs. 2 and 3 for two and three transition samples.* In both these figures, the log

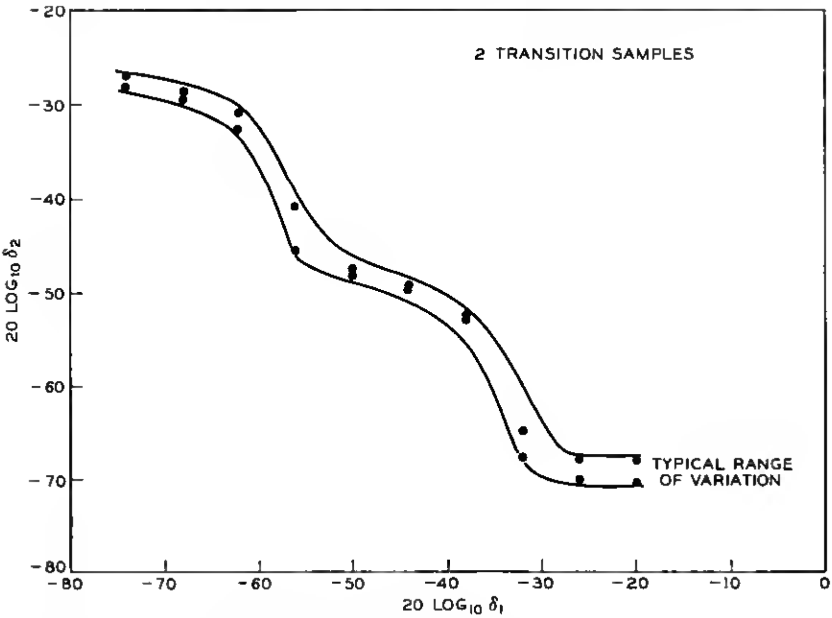

Fig. 2—Tradeoff relations between δ_1 and δ_2 for lowpass frequency sampling filters with two variable transition coefficients.

* The varying nature of the curves of Figs. 2 and 3 is due to the variance in the measured points (heavy dots) as a function of filter bandwidth. Solid curves are shown as an underbound and overbound on the typical behavior of the tradeoff relations.

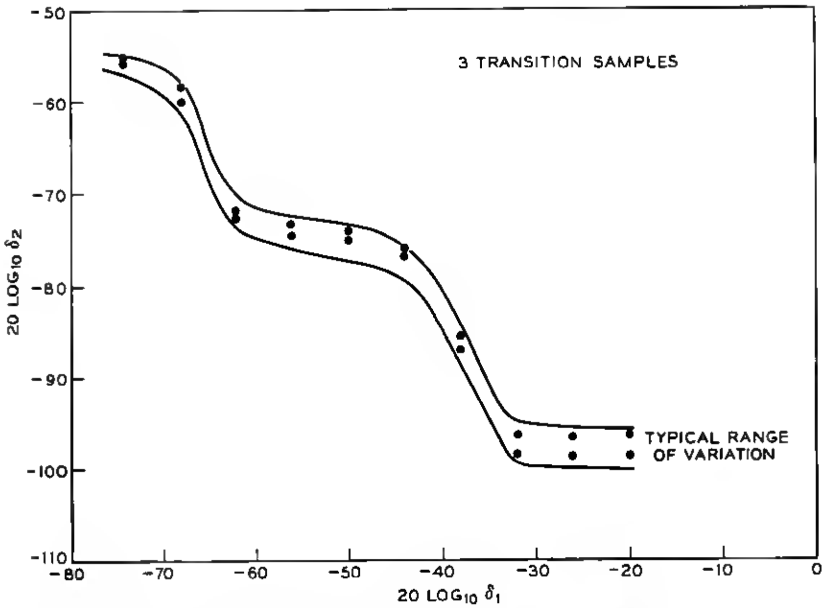

Fig. 3—Tradeoff relations between δ_1 and δ_2 for lowpass frequency sampling filters with three variable transition coefficients.

of out-of-band ripple, δ_2, versus the log of in-band ripple, δ_1, is plotted. Figure 2 shows that for in-band ripples larger than about 0.03 (i.e., 20 $\log_{10} \delta_1$ greater than -30 dB), the out-of-band ripple is in the range -66 to -71 dB. These figures correspond to the cases designed earlier[5] when no constraint on in-band ripple was in effect. At the other extreme of the curve, the out-of-band ripple flattens to between -25 and -30 dB with the in-band ripple, δ_1, in the range 0.0002 to 0.0005 (-74 to -66 dB). The midrange of the curve shows the tradeoff attainable between the two ripples. Figure 3 shows similar results for the case of three transition samples. No simple explanation is available for the general shape of these curves or the differences between the data in Figs. 2 and 3.

Figure 4 shows a comparison between equiripple filters and frequency sampling designs for the specialized case where in-band ripple and out-of-band ripple are equal. In this figure the normalized width of transition band* is plotted as a function of log δ, where δ is the ripple.

* The normalized width of transition band is defined as $D = N[(F_2 - F_1)/F_S]$ where N is the impulse response duration, F_S is the sampling frequency, and F_1 and F_2 are the passband and stopband cutoff frequencies in Hertz.

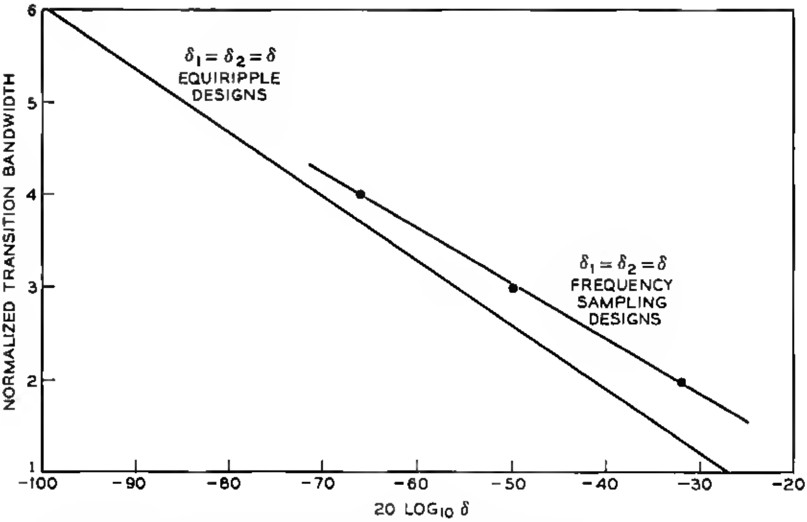

Fig. 4—A comparison between the curves of normalized transition bandwidth versus δ for equiripple filters and frequency sampling filters.

For the frequency sampling designs, the normalized transition widths are 4, 3, and 2 corresponding to 3, 2, and 1 transition samples. At these normalized transition bandwidths the ripple is −66, −50, and −32 dB respectively. The equiripple designs attain the same ripple values at normalized transition bandwidths of approximately 3.7, 2.6, and 1.4. The percentage difference in transition bandwidth for the 3 cases is 8.1, 15.4, and 42.9. Thus, except for the 1 transition point case, the transition bandwidths for frequency sampling designs are reasonably close to the bandwidths for equiripple filters.

VI. DESIGN OF OPTIMAL FILTERS

Just as a few of the DFT coefficients in a transition band could be varied to design reasonably efficient frequency sampling filters, all of the DFT coefficients, or equivalently all of the impulse response coefficients could be varied to give an optimal* approximation to any desired frequency response. Similar optimal approximations have been designed previously using nonlinear optimization procedures[1,2]

* The filters being discussed in this section are optimal in the sense of the theory of Chebyshev approximation on compact sets (i.e., the error of approximation exhibits at least $[(N + 1)/2] + 1$ alternations (of equal amplitude) over the frequency ranges of interest). In most cases, all the peaks of the error function are of the same amplitude, therefore, these filters are often referred to as equiripple filters.

and by polynomial interpolation methods.[3,4] However, the use of linear programming techniques, although significantly slower in running, offers many advantages over other existing design procedures. The design procedure is guaranteed to converge within a fixed number of iterations. Critical frequencies of the desired response can be specified exactly. The programs converge over a very wide range of parameter values. Finally, with the existence and increased understanding of integer linear programming techniques, the design problem can be combined with the coefficient quantization problem to design optimum filters with a prescribed wordlength.

To see how the design of optimal linear phase filters can be accomplished using linear programming techniques, consider the design of a lowpass filter to meet the following set of specifications:

$$\left.\begin{array}{l}\text{Stopband magnitude ripple } \pm\delta_2\\ \text{Passband magnitude ripple } \pm\delta_1\end{array}\right\}\quad \begin{array}{c}\text{minimized or}\\ \text{specified}\end{array}$$

$$\left.\begin{array}{ll}\text{Passband edge} & F_1\\ \text{Stopband edge} & F_2\end{array}\right\}\quad \begin{array}{l}\text{specified}\\ \text{specified}\end{array}$$

$$F_1 < F_2$$

(Phase response is to be linear.)

In this example either δ_1, or δ_2, or some linear combination is minimized. One can also consider the situation where δ_1 and δ_2 are proportionally related (i.e., $\delta_1 = k_1\delta$, $\delta_2 = k_2\delta$ where k_1 and k_2 are constants, and δ is minimized). In this manner a constant ratio between passband and stopband ripple is maintained. Consider the case where δ_1 is specified, and δ_2 is minimized. The linear program which realizes the above specifications can be stated as: find $\{h_n\}$, δ_2 subject to the constraints*:

$$\left.\begin{array}{r}h_0 + 2\sum_{n=1}^{(N-1)/2} h_n \cos \omega n T \leqq 1 + \delta_1\\ -h_0 - 2\sum_{n=1}^{(N-1)/2} h_n \cos \omega n T \leqq -1 + \delta_1\end{array}\right\} 0 \leqq \omega \leqq 2\pi F_1, \tag{15}$$

$$\left.\begin{array}{r}h_0 + 2\sum_{n=1}^{(N-1)/2} h_n \cos \omega n T - \delta_2 \leqq 0\\ -h_0 - 2\sum_{n=1}^{(N-1)/2} h_n \cos \omega n T - \delta_2 \leqq 0\end{array}\right\} 2\pi F_2 \leqq \omega \leqq \pi F_s, \tag{16}$$

$(-\delta_2)$ maximized.

* From this point on, for convenience, we are assuming h_n is defined from $(-(N-1))/2 \leqq n \leqq (N-1)/2$, and is symmetric around $n = 0$. Since N is odd, eq. (8) can be simplified to the form: $H(e^{j\omega T}) = h_0 + \sum_{n=1}^{(N-1)/2} 2h_n \cos \omega n T$.

Before proceeding to typical designs, it is important to note some properties of linear programming problems, and show how they affect the optimal filter design problem. The solution to a linear programming problem of the type shown above with L variables, and M inequality constraints occurs when at least L of the M equations are solved with equality (instead of inequality); the remaining inequalities being met with inequality. For the optimal filter design problem this implies that there are at least L frequencies at which the ripple obtains a maximum. The practical implications of this result are best illustrated in Fig. 5 which shows the frequency response of an equiripple optimal filter with passband ripple δ_1, stopband ripple δ_2, passband edge frequency F_1, and stopband edge frequency F_2. The length of the filter impulse response is N samples. If

N_p = number of ripples in the passband, and
N_s = number of ripples in the stopband,

then

$$N_p + N_s \leqq \frac{(N+1)}{2} \quad (N \quad \text{odd}), \tag{17}$$

since an Nth degree polynomial (the z-transform of the filter impulse response) has at most $(N + 1)/2$ points of zero derivative in the frequency range from 0 to $F_s/2$ Hz. In addition to attaining a maximum

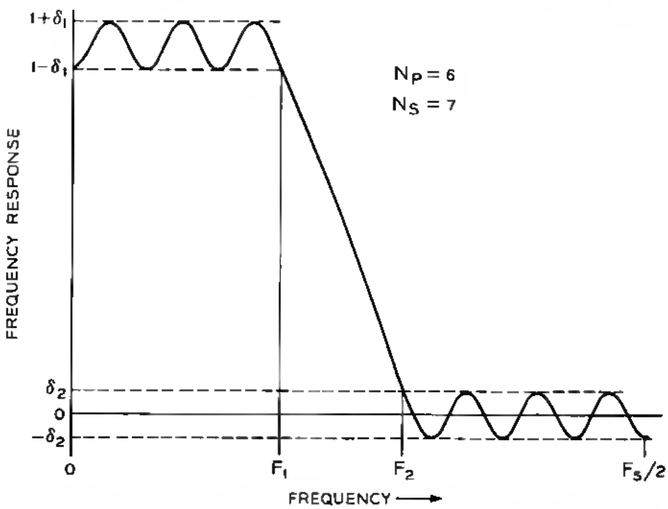

Fig. 5—A typical frequency response for an optimal filter, defining N_p, the number of passband maxima, and N_s the number of stopband maxima.

value at each of the ripple frequencies, the error attains a maximum value at $f = F_1$ and at $f = F_2$ (i.e., at the edges of the transition band). In fact this is how the transition band edges are defined. Thus the number of error maxima, N_e , satisfies the inequality

$$N_e \leqq \frac{(N+1)}{2} + 2. \tag{18}$$

The number of variables N_v in the linear program of eqs. (15) and (16) is

$$N_v = \frac{(N+1)}{2} + 1, \tag{19}$$

where $(N + 1)/2$ coefficients of the impulse response are variable, and one ripple coefficient is variable. Thus eq. (19) shows that the minimum number of error maxima from the linear program solution, although optimal, is one less than the maximum number of error maxima obtainable.* A discussion of the effects of the extra ripple peak on the

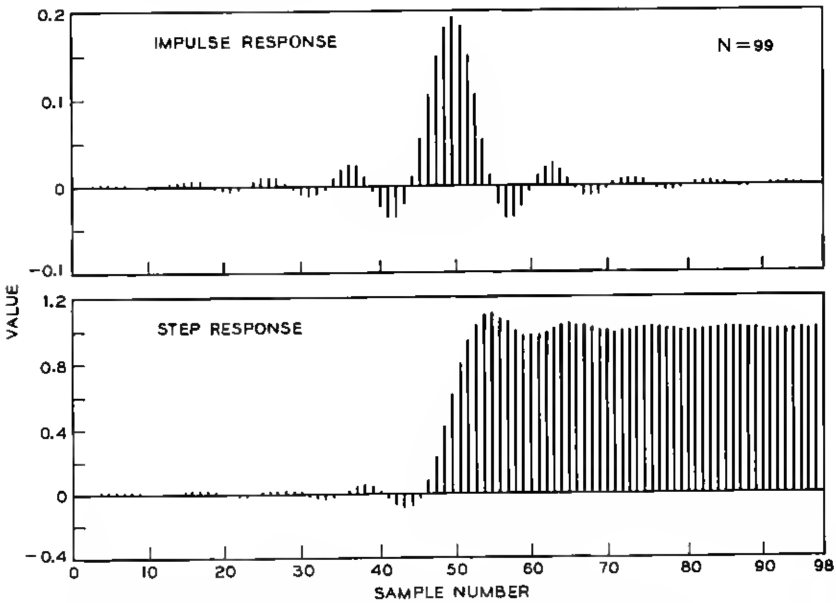

Fig. 6—The impulse and step response for an optimal digital filter with a 99-point impulse response.

* Parks and McClellan[4] have labeled the cases where all the ripples are present as "extra ripple" designs.

width of the transition band is given by Hofstetter, et al.[12] For all practical purposes the loss of the extra ripple is negligible in terms of normalized transition bandwidth, etc. At this point it is worthwhile showing some results of the design procedure.

VII. OPTIMAL FILTER DESIGNS—LOWPASS FILTER EXAMPLES

Using the linear program of eqs. (15) and (16), filters were designed with impulse response durations of up to 99 samples. Figures 6 and 7 show plots of impulse and step responses, and the log magnitude response of a lowpass filter designed from the specifications:

In-band ripple	δ	
Out-of-band ripple	δ	
Passband edge frequency	808	Hz
Stopband edge frequency	1111	Hz
Sampling frequency	10000	Hz.

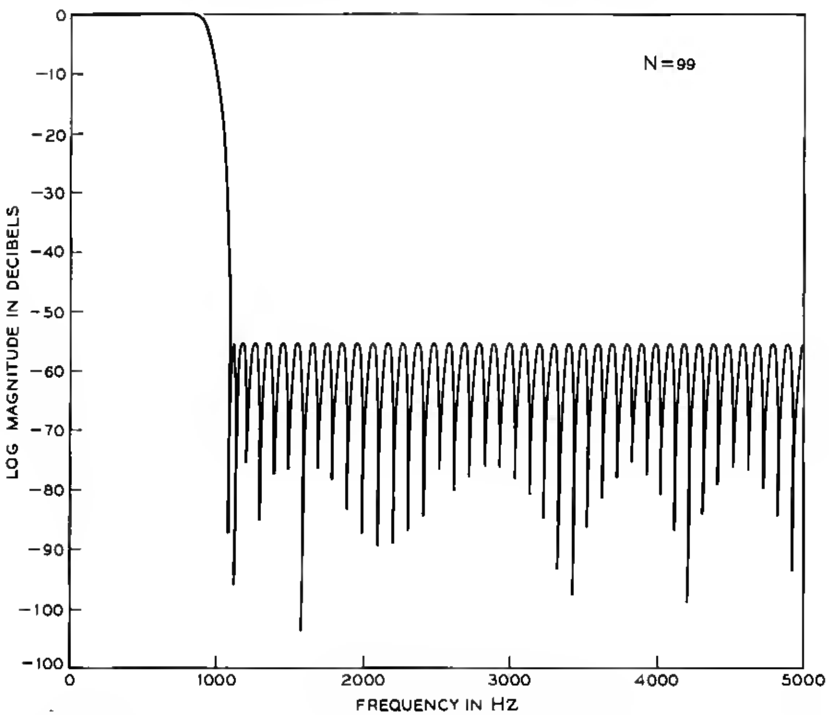

Fig. 7—The frequency response of an optimal digital filter with a 99-point impulse response.

The minimum value of δ, as chosen by the linear program, was $\delta = 0.001724$ or -55.3 dB.

Figure 8 shows a comparison of the normalized transition bandwidth versus log δ_2 for Herrmann–Schuessler equiripple filters with the maximum number of ripples, and the optimal linear program filters. The solid line in this figure shows the Herrmann–Schuessler data for $\delta_1 = \delta_2$, and the data points show the linear program data for several values of N, the impulse response duration. Clearly the differences between the data are insignificant as stated earlier. (The data points which fall below the solid line in Fig. 8 are due to the error in representing the equiripple data by a straight line on these coordinates.)

VIII. OPTIMAL FILTER DESIGNS—OTHER EXAMPLES

As stated earlier, the linear programming technique can design optimal approximations to any desired frequency response. To illustrate this feature several full band differentiators[13] and several filters for use in a digitized version of the A-channel bank[14] (a frequency transmission system in use in the Bell System) were designed.

To design a full band differentiator $H(e^{j\omega T})$ must approximate the normalized response,

$$\hat{H}(e^{j\omega T}) = j\frac{\omega}{\left(\frac{\omega_s}{2}\right)}, \tag{20}$$

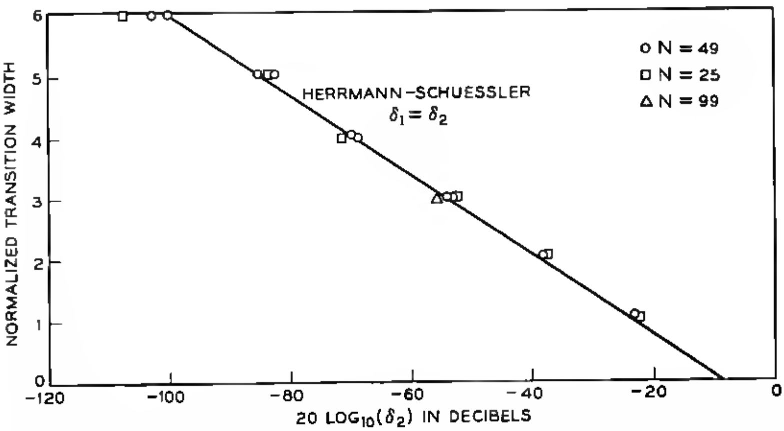

Fig. 8—A comparison between the curves of normalized transition bandwidth versus δ for equiripple filters with the maximum number of ripples, and optimal filters with one ripple omitted. Normalized bandwidth is defined as $D = N(F_2 - F_1)/F_s$.

where $(\omega_s/2)$ is half the radian sampling frequency. To get an optimal error approximation requires,

$$-\delta \leqq |H(e^{j\omega T}) - \hat{H}(e^{j\omega T})| \leqq \delta, \tag{21}$$

where δ is minimized. To get a purely imaginary frequency response as in eq. (20), the impulse response is required to satisfy the symmetry condition,

$$h_n = -h_{N-n-1}, \qquad n = 0, 1, \cdots, \frac{N}{2} - 1, \tag{22}$$

where N is even to take advantage of the noninteger delay.[13] An illustrative example of an $N = 32$ sample differentiator is shown in Fig. 9. This figure shows the impulse response, magnitude response, and the error curve. The peak error, δ, is approximately 0.0057.

One could also consider designing optimal relative error filters by

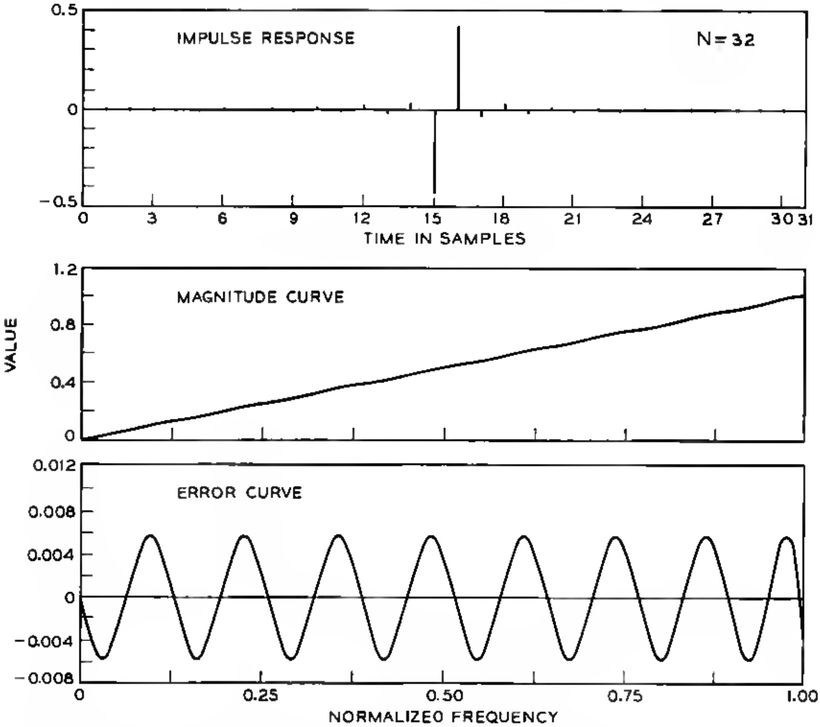

Fig. 9—The impulse response, frequency response, and error curve for a 32-point differentiator with optimal equiripple error.

changing the design equations slightly. For example, to design an optimal relative error differentiator requires,

$$-\delta\omega \leqq | H(e^{j\omega T}) - \hat{H}(e^{j\omega T}) | \leqq \delta\omega \tag{23}$$

(i.e., the envelope of the error in approximation is linear with frequency because the desired frequency response is linear in frequency). An example of an $N = 32$-point differentiator designed in this manner is shown in Fig. 10. The peak error, δ, is now 0.0062, only slightly higher

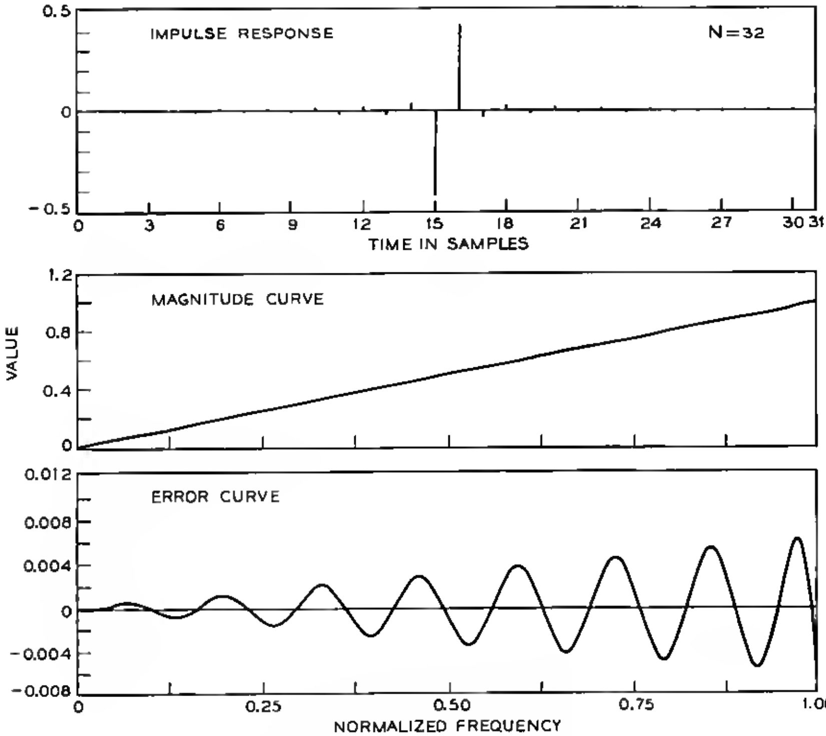

Fig. 10—The impulse response, frequency response, and error curve for a 32-point differentiator with optimal equiripple relative error.

than δ in the optimal error solution. The linearity of the error envelope is evident in Fig. 10.

To illustrate further the versatility of the linear programming approach, a special purpose filter for use in a digital transmission system was designed.[14] The specifications of the filter were:

$F_s = 112$ kHz
Passband ripple, $0 \leqq f \leqq 2$ kHz, $20 \log_{10} (1 + \delta_1) \leqq 0.25$ dB
Stopband ripple, $14 \leqq f \leqq 18$ kHz, $20 \log_{10} \delta_2 \leqq -63$ dB
$30 \leqq f \leqq 34$ kHz, $20 \log_{10} \delta_2 \leqq -63$ dB
$46 \leqq f \leqq 50$ kHz, $20 \log_{10} \delta_2 \leqq -63$ dB

In all other frequency bands, the frequency response was not specified. As an additional constraint on the impulse response, N was chosen arbitrarily to be 21 samples.

Since the filter was completely constrained by the above specifications, it was of interest to see how close the designs could get to the desired specifications. A linear program was written which allowed δ_1 and δ_2 to vary. The results of this program are plotted in Fig. 11. This figure shows a plot of $20 \log_{10} (\delta_2)$ versus $20 \log_{10} [(1 + \delta_1)/(1 - \delta_1)]$ obtained from the program. It also shows a triangle for the desired specifications, and a square for the filter obtained from a manual optimization by S. Freeny. Although none of the filters meets the specifications, the computer optimized designs come much closer than the manual optimization. Figure 12 shows a plot of the filter that comes closest to

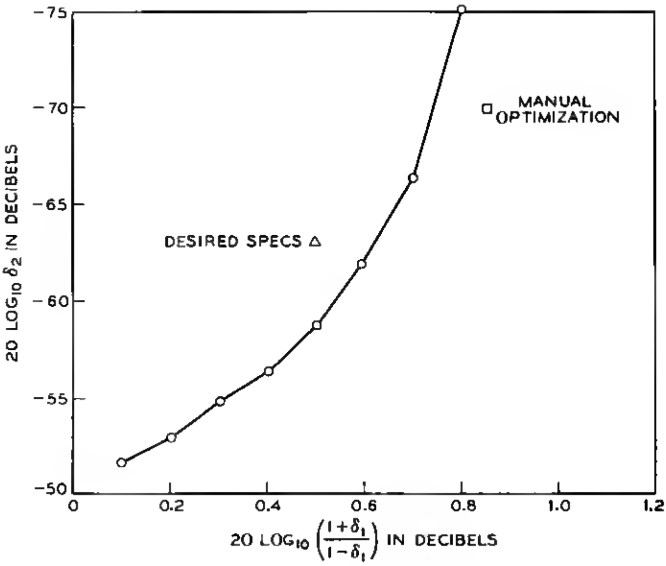

Fig. 11—A plot of the tradeoff relations between $20 \log_{10} ((1 + \delta_1)/(1 - \delta_1))$ and $20 \log_{10} \delta_2$ for a lowpass filter for a digital transmission system.

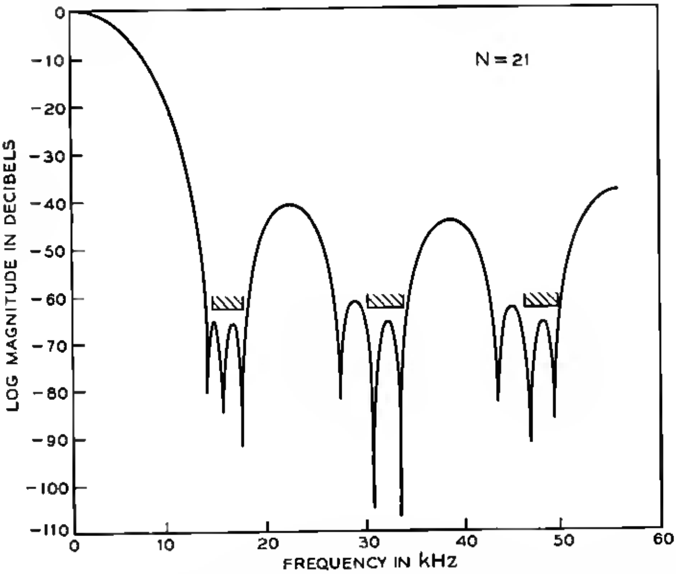

Fig. 12—The frequency response for the closest approximation to the desired specifications for the lowpass filter for the digital transmission system.

the desired specifications. The in-band response differs from specifications by about 0.1 dB, and the out-of-band response meets specifications by over 1 dB. The error is equiripple in each of the out-of-band regions.

IX. DESIGN OF FILTERS WITH SIMULTANEOUS CONSTRAINTS ON THE TIME AND FREQUENCY RESPONSE

The design of digital filters which approximate characteristics of a specified frequency response only has been discussed. Quite often one would like to impose simultaneous restrictions on both the time and frequency response of the filter. For example, in the design of lowpass filters, one would often like to limit the step response overshoot or ripple; at the same time maintaining some reasonable control over the frequency response of the filter. Since the step response is a linear function of the impulse response coefficients, a linear program is capable of setting up constraints of the type discussed above. Consider the design of a lowpass filter (N odd) with specifications:

Passband

$$1 - \delta_1 \leqq h_0 + \sum_{n=1}^{(N-1)/2} 2h_n \cos \omega nT \leqq 1 + \delta_1 , \tag{24}$$

Stopband

$$-\delta_2 \leqq h_0 + \sum_{n=1}^{(N-1)/2} 2h_n \cos \omega nT \leqq \delta_2 , \tag{25}$$

Step Response

$$-\delta_3 \leqq g_n \leqq \delta_3 \qquad n = -\left(\frac{N-1}{2}\right), \cdots , -\left(\frac{N-1}{2}\right) + N_1 , \tag{26}$$

where h_n is the symmetric impulse response of the filter ($h_n = h_{-n}$, $n = 0, 1, \cdots, (N-1)/2$), g_n is the filter step response defined by

$$g_n = \begin{cases} \displaystyle\sum_{m=-(N-1)/2}^{n} h_m & -\left(\dfrac{N-1}{2}\right) \leqq n \leqq \infty \\ 0 & n < -\left(\dfrac{N-1}{2}\right) \end{cases} , \tag{27}$$

and N_1 is the number of samples of the step response being constrained. For optimization there are several alternatives which are possible. One could fix any one or two of the parameters δ_1, δ_2, or δ_3 and minimize the other(s). Alternatively one could set $\delta_1 = \alpha_1\delta$, $\delta_2 = \alpha_2\delta$, and $\delta_3 = \alpha_3\delta$ where α_1, α_2, and α_3 are constants, and simultaneously minimize all three deltas.

To demonstrate this technique, a lowpass filter with $N = 25$ and no constraint on δ_3 was designed. This design is an optimal filter as discussed earlier, and is shown in Fig. 13. In this case δ_1 is set to $25\delta_2$ and the optimization gives $\delta_3 = 0.12$, $\delta_1 = 0.06$, and $\delta_2 = 0.00237$. The results of setting $\delta_3 = 0.03$ and then minimizing the frequency ripple are shown in Fig. 14. The equiripple character of the frequency response has been sacrificed in order to constrain the peak step response ripple. The ripple values for this new design are $\delta_1 = 0.145$ and $\delta_2 = 0.00582$. Using this linear programming technique, one can obtain tradeoffs between any of the deltas to get a design best suited to the particular application. The filter of Fig. 14 was designed for smoothing characteristic speech parameters where step response overshoot is a very important perceptual parameter.

X. DESIGN OF TWO-DIMENSIONAL FIR FILTERS

The techniques of FIR filter design using linear programming are readily extendable to two or more dimensions.[15] Filters of both the frequency sampling type and optimal type have been designed in this manner.

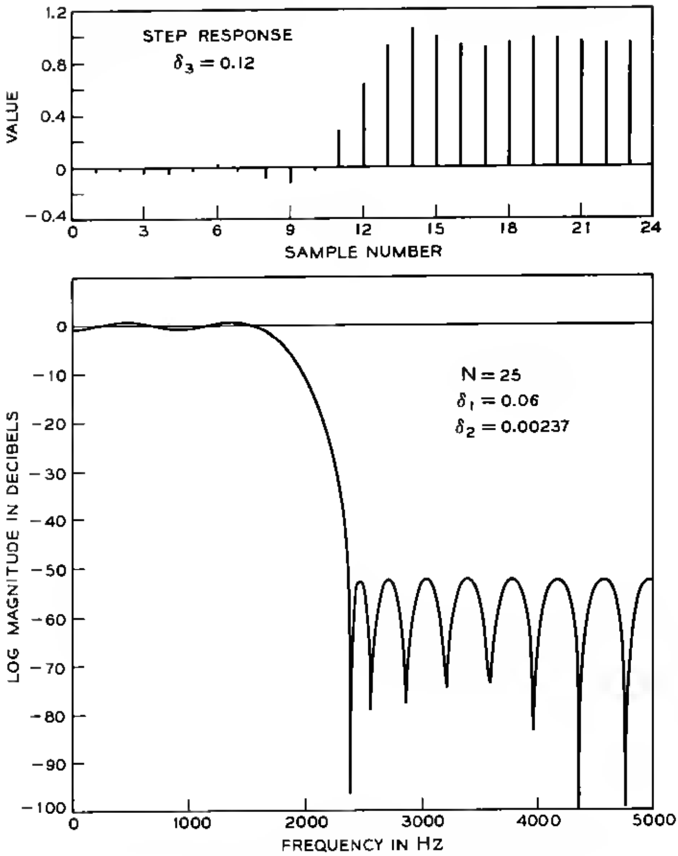

Fig. 13—A plot of the step response and frequency response of an optimal equiripple lowpass filter.

XI. COMPUTATIONAL CONSIDERATIONS

Since one of the major aspects of digital filter design by optimization procedures is the amount of computation necessary to produce a desired result, it is worthwhile discussing some of the details of our simultations.

The programs used throughout this study are APMM,[16] an IBM scientific subroutine which computes a Chebyshev approximation of a given real function over a discrete range, and MINLIN, a program written at Bell Laboratories by Mrs. Wanda Mammel. The running time of these programs is highly dependent on the number of variables, L, the number of inequalities, P, and the "complexity" of the results which determines the number of iterations required to attain a solution.

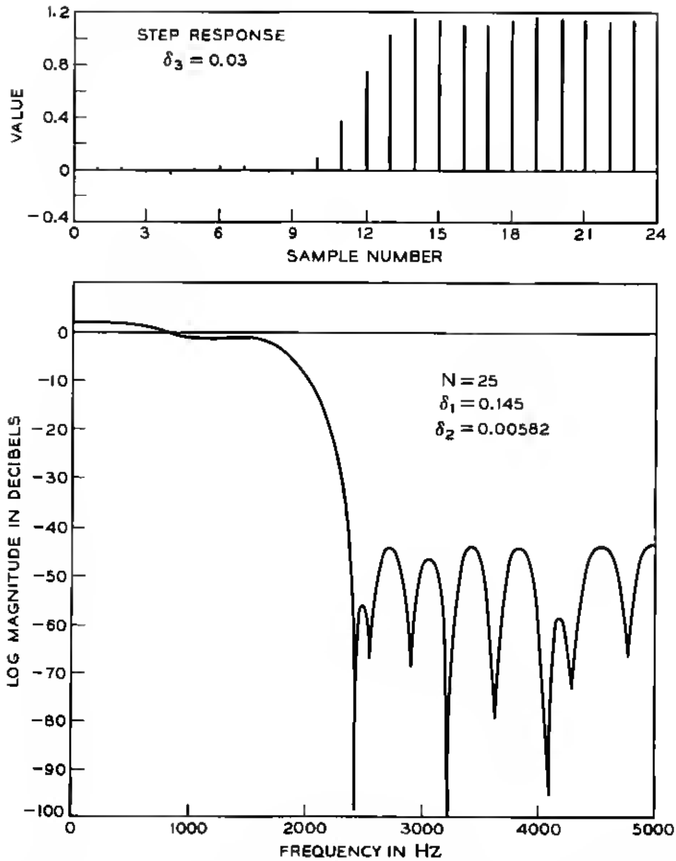

Fig. 14—A plot of the step response and frequency response for a lowpass filter with simultaneous constraints on both the time and frequency response.

The time per iteration is approximately proportional to L^2P. Typical experience indicates that it takes on the order of ten seconds to design the frequency sampling filters discussed earlier (i.e., $L \leqq 3$, P on the order of 1000). The total range of times to design optimal filters using APMM on the Honeywell 645 computer is shown below.

N	No. Iterations	Total Time
25	27 to 58	14 to 47 seconds
49	53 to 82	117 to 194 seconds
99	128	1200 seconds

Although the computation time is reasonably high, it is not impractical

to design high order filters with this technique. The argument can also be made that the most important application of these techniques is in the designs of FIR filters with small values of N (i.e., $N \leqq 50$) in which case the computation time starts becoming more reasonable.

XII. CONCLUSIONS

The design of linear phase FIR digital filters is shown to be a linear programming problem, and many appropriate problems can be solved using this technique. Examples have illustrated several filter areas which are reasonable candidates for linear program designs.

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